Spherically symmetric spacetimes with a trapped surface

Mihalis Dafermos

February 7, 2008

Abstract

This paper investigates the global properties of a class of spherically symmetric spacetimes. The class contains the maximal development of asymptotically flat spherically symmetric initial data for a wide variety of coupled Einstein-matter systems. For this class, it is proven here that the existence of a single trapped or marginally trapped surface implies the completeness of future null infinity and the formation of an event horizon whose area radius is bounded by twice the final Bondi mass.

One of the fundamental questions in gravitational collapse is the so-called *weak cosmic censorship* conjecture [2, 15]. This is the statement that, for generic asymptotically flat initial data, solutions to appropriate Einstein-matter systems possess a complete null infinity. For a precise definition of this latter concept, the reader should consult [2].

In [1], Christodoulou proves weak cosmic censorship for the collapse of a spherically symmetric self-gravitating scalar field. His argument proceeds by showing that data leading to a naked singularity satisfy the property–when perturbed generically–that all singularities are "preceeded" by trapped surfaces. The completeness of null infinity is then inferred from this property.

In the present paper, we formulate general assumptions which, in the context of spherical symmetry, ensure that the existence of a *single* trapped surface suffices to show the formation of a black hole and the completeness of null infinity. The most restrictive assumption excludes a certain kind of TIP not emanating from the center. The assumption has been shown to hold for the maximal development of asymptotically flat data for a wide variety of Einstein-matter systems. Indeed, in this context the assumption corresponds to the statement that "first singularities" arising from non-trapped points can only emanate from the center. For the systems for which the assumptions here hold, the results of this paper suggest a local approach to proving weak cosmic censorship.

Finally, we note that in the process of proving the completeness of null infinity, we obtain an upper bound of twice the final Bondi mass for the area radius of the apparent horizon and–more interestingly–the *event horizon* of the black hole that forms. Upper bounds of this form are commonly known as "Penrose inequalities".

1 First assumptions

Our assumptions are motivated from properties of the maximal developments of spherically symmetric asymptotically flat initial data, for "appropriate" Einstein-matter systems. We will formulate assumptions directly at the level of a two dimensional submanifold $\mathcal{Q}^+$ of 2-dimensional Minkowski space, endowed with metric $-\Omega^2 dudv$, a function r, and a symmetric two-tensor T_{ab}. In applications, $\mathcal{Q}^+$ arises as the quotient of a spherically symmetric maximal development by the group $SO(3)$. In general, we shall include a discussion of the realm of applicability of each assumption–and any subtleties that might arise–immediately after its formulation.

1.1 The quotient manifold

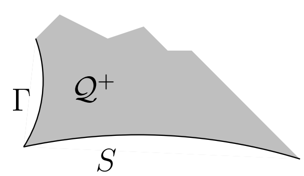
Let $\mathbf{R}^2$ denote the standard plane. We will call its coordinates (u, v), and we will depict the v-axis at 45 degrees from the horizontal, and the u-axis at 135 degrees. Unless otherwise noted, causal-geometric concepts, like the word "timelike" or the set $J^+(p)$, etc., will refer to the metric $-dudv$ of $\mathbf{R}^2$, future oriented in the standard way so that u and v are both increasing towards the future. Our first assumption is

A′ We are given a bounded two-dimensional submanifold $\mathcal{Q}^+ \subset \mathbf{R}^2$ with boundary $\Gamma \cup S$, where Γ is a connected timelike curve, and S is a connected spacelike curve, and $\Gamma \cap S$ is a single point p. We assume that on $\mathcal{Q}^+$ we are given C^1 functions r, Ω, such that $\Omega > 0$, $r \geq 0$, and $r(q) = 0$ iff $q \in \Gamma$. Defining the so-called *Hawking mass*:

$$m = \frac{r}{2}(1 + 4\Omega^{-2}\partial_u r \partial_v r),$$

we assume that m is uniformly bounded along S.

We assume that $\mathcal{Q}^+$ is foliated by connected constant v-segments with past endpoint on S, and also by connected constant u-segments with past endpoint on $\Gamma \cup S$. We will call the former "ingoing" segments, and the "latter" outgoing.



Given an "appropriate" notion of spherically symmetric asymptotically flat dataset $(\Sigma, \bar{g}, K, \ldots)$ for a "reasonable" Einstein-matter system, with one end, it follows that the maximal Cauchy development[1] $(\mathcal{M}, g)$ will admit an isometry

[1] "reasonable" means in particular that this concept can be defined with its standard properties...

action by the group $SO(3)$ and moreover, $\mathcal{Q} = \mathcal{M}/SO(3)$ will inherit[2] the structure of a Lorentzian manifold. If $\mathcal{Q}^+$ denotes the quotient of $\mathcal{M} \cap J_g^+(\Sigma)$, then $\mathcal{Q}^+$ can be conformally embedded into $\mathbf{R}^2$ so as to satisfy the conditions of $\mathbf{A}'$, where $-\Omega^2 dudv$ represents the metric of $\mathcal{Q}$, and $-\Omega^2 dudv + r^2\gamma$ represents the metric of $\mathcal{M}$, where γ is the standard metric on S^2. Γ, the center, corresponds to the set of fixed points of the group action.

1.2 Structure equations

Our next assumption is

B′ We have a symmetric 2-tensor, with bounded components T_{uu}, T_{uv}, T_{vv}, defined on $\mathcal{Q}^+$. The following equations hold pointwise almost everywhere:

$$\partial_u(\Omega^{-2}\partial_u r) = -4\pi r \Omega^{-2} T_{uu} \tag{1}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -4\pi r \Omega^{-2} T_{vv}, \tag{2}$$

$$\partial_u m = 8\pi r^2 \Omega^{-2}(T_{uv}\partial_u r - T_{uu}\partial_v r), \tag{3}$$

$$\partial_v m = 8\pi r^2 \Omega^{-2}(T_{uv}\partial_v r - T_{vv}\partial_u r). \tag{4}$$

If the metric g defined by $-\Omega^2 dudv + r^2\gamma$ is C^2, then equations (1)–(4) are just identities necessarily satsified by the uu, uv and vv components of the tensor

$$T_{\mu\nu} = \frac{1}{8\pi}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R).$$

In the context of spherical symmetry, one often defines less regular notions of solutions tied explicitly to (1)–(4). It is for this reason that we have prefered to postulate (1)–(4) directly.

1.3 Positive energy condition

Γ′ We have

$$T_{uu} \geq 0, T_{vv} \geq 0, T_{uv} \geq 0 \tag{5}$$

Again, when $T_{\mu\nu}$ is the energy momentum tensor of matter, coupled to gravity via the Einstein equations, the above assupmtion is precisely the positive energy condition.

1.4 No anti-trapped surfaces initially

Δ′ We have

$$\partial_u r < 0$$

along S.

This condition has been introduced by Christodoulou in [3]. It is motivated, in part, by Proposition 1 of the next section.

[2] "appropriate" is taken to ensure this...

2 $\mathcal{R}$,$\mathcal{T}$, and $\mathcal{A}$

In what follows we assume $\mathbf{A}'$–$\mathbf{\Delta}'$.

Following the notation of [3], we define the *regular* or *non-trapped region*

$$\mathcal{R} = \{p \in \mathcal{Q}^+ \text{ such that } \partial_v r > 0, \partial_u r < 0\},$$

the *trapped region*

$$\mathcal{T} = \{p \in \mathcal{Q}^+ \text{ such that } \partial_v r < 0, \partial_u r < 0\},$$

and the *marginally trapped region*

$$\mathcal{A} = \{p \in \mathcal{Q}^+ \text{ such that } \partial_v r = 0, \partial_u r < 0\}.$$

We include for completeness the proof of the following proposition, due to Christodoulou [3]:

Proposition 1 *We have $\mathcal{Q}^+ = \mathcal{R} \cup \mathcal{T} \cup \mathcal{A}$, i.e. anti-trapped surfaces are non-evolutionary. On $\mathcal{A}$, $1 - \frac{2m}{r} = 0$, while $1 - \frac{2m}{r} < 0$ in $\mathcal{T}$. In $\mathcal{R}$, we have $m \geq 0$ and $\partial_v m \geq 0$, $\partial_u m \leq 0$. Moreover, if $(u, v) \in \mathcal{T}$, then $(u, v^*) \in \mathcal{T}$ for $v^* > v$, and similarly, if $(u, v) \in \mathcal{T} \cup \mathcal{A}$, then $(u, v^*) \in \mathcal{T} \cup \mathcal{A}$.*

Proof. By assumption $\mathbf{A}'$, all ingoing curves in $\mathcal{Q}$ have past endpoint on S. Thus, integrating (1) from S, we obtain–in view of assumptions $\mathbf{\Gamma}'$ and $\mathbf{\Delta}'$–that $\Omega^{-2}\partial_u r < 0$, and thus $\partial_u r < 0$ in $\mathcal{Q}^+$. This proves the first statement.

The second statement is an immediate consequence of the identity

$$1 - \frac{2m}{r} = -\frac{4}{\Omega^2}\partial_u r \partial_v r,$$

in view of the inequality $\partial_u r < 0$.

Integrating now (2) yields that $\Omega^{-2}\partial_v r$ is a nonincreasing function of v, and this immediately yields the final statement.

For the third statement, note first that the inequalities $\partial_u m \leq 0$, $\partial_v m \geq 0$, on $\mathcal{R}$, are trivial consequences of the signs of $\partial_v r$ and $\partial_u r$ in (3) and (4), in view of Assumption $\mathbf{\Gamma}'$. To show that $m \geq 0$ on $\mathcal{R}$, in view of the fact that the statement proved in the previous paragraph shows that $(u, v) \in \mathcal{R}$ implies $(u, v^*) \in \mathcal{R}$ for all $v^* \leq v$, it suffices to show that $m \geq 0$ on $(\Gamma \cup S) \cap \mathcal{R}$.

The condition $m = 0$ on Γ is implied by the regularity assumption of $\mathbf{A}'$. Let $\mathbf{K}$ denote the unit tangent vector on S such that $\mathbf{K} \cdot v > 0$. It follows from (4), (3), that $\mathbf{K} \cdot m \geq 0$ on $S \cap \mathcal{R}$. Let s denote the coordinate on S with $\mathbf{K} \cdot s = 1$, $s = 0$ at $\Gamma \cap S$. If $s' \in S \cap \mathcal{R}$ then either $[0, s') \in S \cap \mathcal{R}$, in which case $m(s') \geq m(0) = 0$, or else $(t', s') \subset \mathcal{R}$ for $t' \in \mathcal{A}$, in which case $m(s') \geq m(t')$. But $m(t') = \frac{r(t')}{2} > 0$. This completes the proof. □

3 Null infinity

The curve S acquires a unique limit point i_0 in $\overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+$ called *spacelike infinity*. Let $\mathcal{U}$ denote the set of all u defined by

$$\mathcal{U} = \left\{ u : \sup_{v:(u,v)\in\mathcal{Q}^+} r(u,v) = \infty \right\}.$$

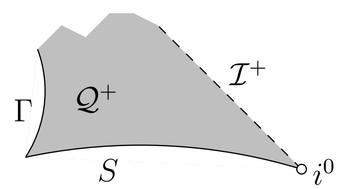
This set may of course be empty, even if $r \to \infty$ along S. For each $u \in \mathcal{U}$, there is clearly a unique $v^*(u)$ such that

$$(u, v^*(u)) \in \overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+.$$

Define

$$\mathcal{I}^+ = \bigcup_{u\in\mathcal{U}} (u, v^*(u))$$

We will call $\mathcal{I}^+$ *future null infinity*.



We have the following

Proposition 2 *If non-empty, $\mathcal{I}^+$ is a connected ingoing null ray with past limit point i_0.*

Proof. Let $i_0 = (U, V)$. If $v_0 < V$, then since $\partial_u r < 0$ in $\mathcal{Q}^+$, we have an a priori bound for r in $\mathcal{Q}^+ \cap \{v \le v_0\}$ by the supremum of r on $\{v \le v_0\} \cap S$. Thus $\mathcal{I}^+ \cap \{v = v_0\} = \emptyset$, i.e., $\mathcal{I}^+ \subset \{v = V\}$.

Suppose now that $(u_0, V) \in \mathcal{I}^+$ and let $u < u_0$. Since, by definition $\lim_{v\to V} r(u_0, v) = \infty$, while on the other hand $r(u,v) > r(u_0,v)$ by the inequality $\partial_u r < 0$, it follows that $\lim_{v\to\infty} r(u,v) = \infty$, i.e., $(u,V) \in \mathcal{I}^+$. This proves the proposition. $\Box$

We introduce the assumption:

E′ $\mathcal{I}^+$ is non-empty.

In applications to the initial value problem, for initial data such that the matter is of compact support (or electrovacuum outside a compact set)–and such that the cosmological constant vanishes!–this assumption is immediate by Birkhoff's theorem and the domain of dependence property. It can also be reasonably expected to hold for matter whose initial asymptotic behavior is sufficiently tame.

The set $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+$ is the so-called *domain of outer communications.* Clearly, by Proposition 1, it follows that

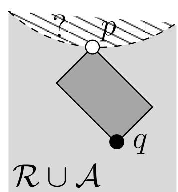
$$J^-(\mathcal{I}^+) \cap \mathcal{Q}^+ \subset \mathcal{R}. \tag{6}$$

From the inequalities $\partial_v r \geq 0$, $\partial_u r \leq 0$ in $\mathcal{R}$, and its uniform boundedness intially by $\mathbf{A}'$, it is clear that m extends to a nonincreasing non-negative function along $\mathcal{I}^+$. We will denote $\inf_{\mathcal{I}^+} m$ by M_f, and refer to this as the *final Bondi mass.*

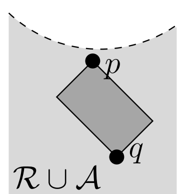
4 The extension principle

To proceed further we will need one final assumption:

$\mathbf{\Sigma T}'$ Let $p \in \overline{\mathcal{R}} \backslash \overline{\Gamma}$, and $q \in \overline{\mathcal{R}} \cap I^-(p)$ such that such that $J^-(p) \cap J^+(q) \backslash \{p\} \subset \mathcal{R} \cup \mathcal{A}$:



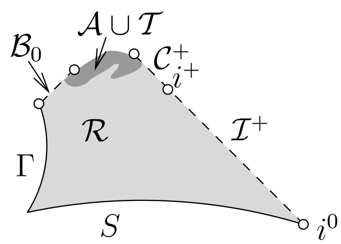
Then $p \in \mathcal{R} \cup \mathcal{A}$.



In the evolutionary context, this assumption can be stated informally as the proposition that a "first singularity" emanating from the regular region can only arise from the center. It has been proven to hold for a wide class of self-gravitating Higgs' fields [10] and for self-gravitating collisionless matter [16, 17, 11]. It can reasonably be expected to hold for charged scalar fields, self-gravitating sigma models, Yang-Mills fields, and more complicated systems arising from the coupling of all the aforementioned. The reader should note, however, that, as applied to maximal developments, $\mathbf{\Sigma T}'$ is more restrictive than the previous assumptions, as it captures a non-trivial feature of the behavior of solutions to the p.d.e., and not just "general theory". For instance, $\mathbf{\Sigma T}'$ is violated for a self-gravitating dust.

There are various equivalent ways of formulating $\mathbf{\Sigma T}'$. Let $\mathcal{Q}^*$ denote the intersection of $\overline{\mathcal{Q}^+}$ with the set $\{v \neq V\}$. Since r is decreasing on ingoing null rays, r can be extended by monotonicity to a function defined on $\mathcal{Q}^*$. In view of the fact that a p satisfying $\mathbf{\Sigma T}'$ is necessarily in $\mathcal{Q}^*$, one can replace the assumption $p \notin \overline{\Gamma}$ with the assumption $p \in \mathcal{Q}^*$ and $r(p) > 0$.

Assuming $\mathbf{\Sigma T'}$, we can proceed to describe $\overline{\mathcal{R}} \setminus \mathcal{Q}^+$. One can easily deduce from $\mathbf{\Sigma T'}$ and Proposition 1 that if $p \in \overline{\mathcal{R}} \setminus \mathcal{Q}^+$, then either p is on the outgoing null ray emanating from the future limit point of Γ, or on the ingoing null ray emanating from the future limit point, call it i^+, of $\mathcal{I}^+$.[3] Let us denote by $\mathcal{B}_0$ the former ray, intersected with $\mathcal{Q}^*$, and by $\mathcal{C}^+$, the latter ray intersected with $\overline{\mathcal{Q}}^+$, where this latter ray is taken *not* to include i^+. The set $\overline{\mathcal{R}} \setminus \mathcal{Q}^+$ is then the union of $\mathcal{I}^+$, i^+, and connected closed subsets of $\mathcal{B}_0$ and $\mathcal{C}^+$.



Of course, $\mathcal{B}_0$ may be a single point, and $\mathcal{C}^+$ may indeed be empty. Moreover, i^+ and $\mathcal{B}_0$ may coincide, cf. Minkowski space.

5 The completeness of null infinity

In what follows, we assume $\mathbf{A'}$–$\mathbf{\Sigma T'}$. The main result of this paper is

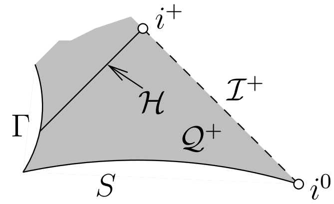
Theorem 1 *If $\mathcal{A}$ is non-empty, then $\mathcal{I}^+$ is future complete.*

By the statement that "$\mathcal{I}^+$ is future complete", we mean that the spacetime $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+ \times S^2$ with respect to the metric $-\Omega^2 du dv + r^2\gamma$ satisfies the formulation of Christodoulou described in [2].

Proof. Since $\mathcal{A} \cup \mathcal{T}$ is non-empty, it follows by (6) that $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+$ has a future boundary in $\mathcal{Q}^+$. This future boundary is an outgoing null ray $\mathcal{H}$ we shall call the *event horizon.* It is clear that

$$\mathcal{H} \subset \mathcal{R} \cup \mathcal{A}. \tag{7}$$

Recall $\mathcal{Q}^*$ defined earlier. By $\mathbf{\Sigma T'}$, we have $\mathcal{H} = \mathcal{Q}^* \cap \{u = \tilde{U}\}$ for some $\tilde{U}$, i.e., $\mathcal{H}$ cannot terminate before reaching i^+.



[3] Note that *a priori*, as defined, i^+ may or may not be contained in $\mathcal{I}^+$.

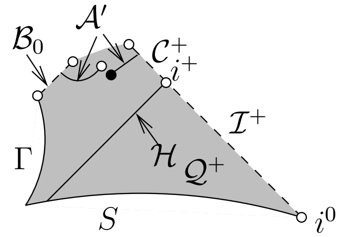
5.1 Outermost apparent horizon

Define the set

$$\mathcal{A}' = \{(u,v) \in \mathcal{A} : (u^*,v) \in \mathcal{R} \text{ for all } u^* < u\}.$$

We will call this set the *outermost apparent horizon.* First we show

Lemma 1 *$\mathcal{A}'$ is a non-empty (not necessarily connected) achronal curve intersecting all ingoing null curves for $v \geq v_0$, for sufficiently large v_0.*



Proof. The statement that $\mathcal{A}'$ is achronal is immediate in view of its definition and Proposition 1. Let $(u',v') \in \mathcal{A} \cup \mathcal{T}$. Let $v_0 > v'$ be such that $S \cap \{v \geq v_0\} \subset \mathcal{R}$, and consider an ingoing null ray $v = v''$ emanating from a point $(u'',v'') \in S$, where $v'' \geq v_0$. If $[u'',u'] \times v'' \in \mathcal{Q}$, then by Proposition 1, it follows that $(u',v'') \in \mathcal{T} \cup \mathcal{A}$, and thus, there must exist a point $(u^*,v'') \in \mathcal{A}'$. In general, if $(\tilde{u},v'') \in \mathcal{T} \cup \mathcal{A}$, then again there must exist a point $(u^*,v'') \in \mathcal{A}'$. Thus, we must exclude the possibility that $[u',\bar{u}) \times v'' \in \mathcal{R}$ for some $\bar{u}$, with $(\bar{u},v'') \in \overline{\mathcal{R}} \cap \mathcal{Q}^* \setminus \mathcal{Q}$. But this would imply that $(u^*,v^*) \in \mathcal{R}$ for all $u^* < \bar{u}$, $v^* \leq v''$. Since $\bar{u} \leq u'$, it follows that the outgoing null curve $u = \bar{u}$ intersects $\mathcal{R} \cup \mathcal{A}$. In particular, there is a point on this curve such that $r \geq c > 0$. Thus, it follows from $\partial_v r \geq 0$ in $\mathcal{R} \cup \mathcal{A}$ that $r(\bar{u},v'') \geq c$, and thus, by $\mathbf{\Sigma T}'$, $(\bar{u},v'') \in \mathcal{Q}$, a contradiction. $\Box$

Note that if $\mathcal{C}^+ \cap \mathcal{B}_0 \neq \emptyset$, then it is *not* necessarily the case that arbitrarily late outgoing null curves enter $\mathcal{A} \cup \mathcal{T}$.

5.2 Penrose inequality for the apparent horizon

We have in fact the following

Lemma 2 *On $\mathcal{A}'$, $r \leq 2M_f$.*

The above lemma is one manifestation of what is commonly referred to in the literature as a *Penrose inequality.*[4] In particular, this implies that M_f is strictly positive.

Proof. Let $M > M_f$. There exists a point $(u_0,V) \in \mathcal{I}^+$ such that $m(u_0,V) \leq M$. If $(u',v') \in \mathcal{A}'$, we have

$$(u',u_0] \times v' \cup u_0 \times [v',V) \subset \mathcal{R}.$$

[4]The term *Penrose inequality* has also been applied to describe similar bounds for the mass at spacelike infinity. See [13].

In particular, integrating the inequalities $\partial_v m \geq 0$, $\partial_u m \leq 0$, along these segments, yields

$$m(u', v') \leq M.$$

But, $m(u', v') = \frac{r}{2}(u', v')$, since $(u', v') \in \mathcal{A}'$, thus

$$r \leq 2M.$$

Since this is true for all $M > M_f$, $r \leq 2M_f$. $\square$

5.3 Penrose inequality for the event horizon

Next, we shall prove the following:

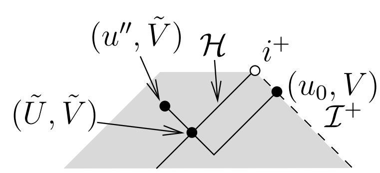
Lemma 3 *On $\mathcal{H}$, $r \leq 2M_f$.*

The above lemma is yet another manifestation of a *Penrose inequality*. Since $\partial_u r < 0$ immediately yields

$$\sup_{\mathcal{H}} r \geq \sup_{\mathcal{A}'} r,$$

it follows that the above lemma, interpreted as a lower bound on M_f, is a stronger statement than the previous.

Proof. Suppose not. Then there exists a point $(\tilde{U}, \tilde{V})$ on the event horizon such that $r(\tilde{U}, \tilde{V}) = R > 2M_f$. Let M satisfy $\frac{R}{2} > M > M_f$. It follows that there exists a point $(u_0, V) \in \mathcal{I}^+$ such that $m(u_0, V) \leq M$.



Thus, by the inequalities $\partial_v m \geq 0$, $\partial_u m \leq 0$ in $J^-(\mathcal{I}^+) \cup \mathcal{H}$, it follows that $m \leq M$ in $(J^-(\mathcal{I}^+) \cup \mathcal{H}) \cap \{u \geq u^0\}$, in particular, on $\mathcal{H}$.

It follows now from (7) that $r(\tilde{U}, v^*) \geq R$, for $v^* \geq \tilde{V}$. In particular, since $m(\tilde{U}, v^*) \leq M$, it follows that $1 - \frac{2m}{r} > 0$ on $\mathcal{H} \cap \{v \geq \tilde{V}\}$. By Proposition 1, we have in fact $\mathcal{H} \subset \mathcal{R}$.

By continuity, it now follows that there exists a u'', such that $[\tilde{U}, u''] \times \tilde{V} \subset \mathcal{R}$, and $r(u^*, \tilde{V}) > R'$, for some $R' < R$ with $1 - \frac{2M}{R'} > 0$, for $u^* \in [\tilde{U}, u'']$. Consider the set

$$\begin{aligned} X &= [\tilde{U}, u''] \times [\tilde{V}, V) \cap \mathcal{Q}^+ \\ &\quad \cap \{(\bar{u}, \bar{v}) : r(\tilde{u}, \tilde{v}) > R'', m(\tilde{u}, \tilde{v}) < M' \text{ for all } \tilde{U} \leq \tilde{u} \leq \bar{u}, \tilde{V} \leq \tilde{v} \leq \bar{v}\} \end{aligned}$$

for some $M' > M$, $R'' < R'$ such that $1 - \frac{2M'}{R''} > 0$. By the continuity of r and m, X is clearly an open subset of $[\tilde{U}, u''] \times [\tilde{V}, V) \cap \mathcal{Q}^+$ in the latter set's topology. Moreover, since $1 - \frac{2m}{r} > 0$ in X, it follows that $X \subset \mathcal{R}$. Since this

implies that $\partial_v r \geq 0$, $\partial_u m \leq 0$, it follows that $r(\bar{u},\bar{v}) > R'$, $m(\bar{u},\bar{v}) \leq M$, and, thus X is closed. Since X is clearly connected, it follows that

$$X = [\tilde{U}, u''] \times [\tilde{V}, V) \cap \mathcal{Q}^+.$$

It follows that $r \geq R' > 0$ on $[\tilde{U}, u''] \times [v, V) \cap \mathcal{Q}^*$, and thus, by $\mathbf{\Sigma T'}$,

$$[\tilde{U}, u''] \times [\tilde{V}, V) \cap \mathcal{Q}^* = [\tilde{U}, u''] \times [\tilde{V}, V) \cap \mathcal{Q}^+.$$

But the left hand side is the closure of the right hand side in the topology of $[\tilde{U}, u''] \times [\tilde{V}, V)$. Thus,

$$[\tilde{U}, u''] \times [\tilde{V}, V) \cap \mathcal{Q}^+$$

is an open and closed subset of

$$[\tilde{U}, u''] \times [\tilde{V}, V),$$

and consequently,

$$[\tilde{U}, u''] \times [\tilde{V}, V) \cap \mathcal{Q}^+ = [\tilde{U}, u''] \times [\tilde{V}, V),$$

and moreover,

$$[u_0, u''] \times [\tilde{V}, V) \subset \mathcal{R}.$$

Integrating (3), noting that both terms on the right hand side are non-negative in $\mathcal{R}$, and that the Hawking mass satisfies $0 \leq m \leq M$ in $[u_0, u''] \times [\tilde{V}, V)$, we obtain the estimate

$$\sup_{\bar{v} \geq \tilde{V}} \int_{u_0}^{u^*} 8\pi r^2 T_{uu} \frac{1 - \frac{2m}{r}}{(-\partial_u r)}(\bar{u},\bar{v}) d\bar{u} \leq M. \tag{8}$$

Consider now the quantity

$$\frac{\partial_v r}{1 - \frac{2m}{r}}.$$

This is well defined at (u_0, v^*) for all $v_* \in [\tilde{V}, V)$. We easily compute the identity

$$\partial_u \frac{\partial_v r}{1 - \frac{2m}{r}} = \frac{4\pi r T_{uu}}{\partial_u r} \frac{\partial_v r}{1 - \frac{2m}{r}}. \tag{9}$$

In view of the bounds

$$r \geq R',$$

$$1 - \frac{2m}{r} \geq 1 - \frac{2M}{R'}$$

in $[\tilde{U}, u''] \times [\tilde{V}, V)$, (8) yields

$$\sup_{\bar{v} \geq \tilde{V}} \int_{u_0}^{u^*} \frac{4\pi r T_{uu}}{(-\partial_u r)} d\bar{u} \leq \frac{M}{2(R' - 2M)} \tag{10}$$

and thus, integrating (9),

$$\frac{\partial_v r}{1-\frac{2m}{r}}(u^*,v^*) \geq \exp(-M(2(R'-2M))^{-1})\frac{\partial_v r}{1-\frac{2m}{r}}(u_0,v^*).$$

We obtain immediately that

$$\partial_v r(u^*,v^*) \geq \left(1-\frac{2M'}{R}\right)^{-1} \exp(-M(2(R'-2M))^{-1})\partial_v r(u_0,v^*),$$

and thus, upon integration $r(u^*,v^*) \to \infty$ as $v^* \to V$, since $r(u_0,v^*) \to \infty$, i.e. $(u^*,V) \in \mathcal{I}^+$. It follows that $\mathcal{H}$ is not the event horizon after all, a contradiction. $\Box$

5.4 Christodoulou's completeness condition

The completeness statement we shall prove is the formulation of [2]. In our context, this takes the following form: Fix an outgoing null ray $u = u_0$, for $u_0 < \tilde{U}$, and consider the vector field

$$\mathbf{X}(u,v) = \frac{\partial_v r(u,v)(1-\frac{2m}{r})(u_0,v)\partial_u r(u,v)}{\partial_v r(u_0,v)(1-\frac{2m}{r})(u,v)\partial_u r(u_0,v)}\frac{\partial}{\partial \mathbf{u}}$$

on $J^-(\mathcal{I}^+)\cap\mathcal{Q}^+$. This vector field is parallel along the outgoing null ray $u = u_0$, and on all ingoing null rays. We shall show that the affine length

$$\int_{u_0}^{\tilde{U}} \mathbf{X}(u,v)\cdot u du \to \infty$$

as $v \to \infty$.

Let $R > 2M_f$, and consider the curve $\{r = R\} \cap J^-(\mathcal{I}^+)$. By Lemma 1, for sufficiently large $v_0 < V$, all ingoing null curves with $v \geq v_0$ intersect $\{r = R\} \cap J^-(\mathcal{I}^+)$ at a unique point $(u^*(v), v)$, depending on v. Let M denote the Bondi mass at u_0. We have

$$\begin{aligned}
\int_{u_0}^{\tilde{U}} \mathbf{X}(u,v)\cdot u du &\geq \int_{u_0}^{u^*(v)} \mathbf{X}(u,v)\cdot u du \\
&\geq \frac{1}{(-\partial_u r)(u_0,v)}\int_{u_0}^{u^*(v)} \exp\left(\int_{u_0}^{u} \frac{4\pi r T_{uu}}{\partial_u r}(\bar{u},v)d\bar{u}\right)(-\partial_u r)du \\
&\geq \frac{(r(u_0,v)-R)}{(-\partial_u r)(u_0,v)}\exp(-M(2(R-2M))^{-1}), \qquad (11)
\end{aligned}$$

where (11) follows from the bound

$$\int_{u_0}^{u} \frac{4\pi r T_{uu}}{(-\partial_u r)}(\bar{u},v)d\bar{u} \leq \frac{M}{2(R-2M)},$$

which is proven as in (10). Since $r(u_0, v) \to \infty$ as $v \to \infty$, to show the theorem, it suffices to show that $(-\partial_u r)(u_0, v)$ is uniformly bounded in v.

Consider the quantity

$$\frac{\partial_u r}{1 - \frac{2m}{r}}.$$

In analogy to (9), we have

$$\partial_v \frac{\partial_u r}{1 - \frac{2m}{r}} = \frac{4\pi r T_{vv}}{\partial_v r} \frac{\partial_u r}{1 - \frac{2m}{r}},$$

and thus

$$\frac{\partial_u r}{1 - \frac{2m}{r}}(u_0, v) = \exp\left(\int_{v_0}^{v} \frac{4\pi r T_{vv}}{\partial_v r}(u_0, \bar{v}) d\bar{v}\right) \frac{\partial_u r}{1 - \frac{2m}{r}}(u_0, v).$$

Recalling the defintion of M, we can choose v_0 such that $1 - \frac{r(u_0, v_0)}{2M} > 0$. Set $R' = r(u_0, v_0)$. Then, as in (10), we have

$$\int_{v_0}^{v} \frac{4\pi r T_{vv}}{(\partial_v r)} d\bar{v} \leq \frac{M}{2(R' - 2M)},$$

and thus,

$$-\partial_u r(u_0, v) \leq \left(1 - \frac{R'}{2M}\right)^{-1} \exp\left(M(2(R' - 2M))^{-1}\right)$$

for $v \geq v_0$. This completes the proof. □

6 Remarks

It is clear that in the above theorem, we only used the condition $\mathcal{A} \cup \mathcal{T} \neq \emptyset$ to infer $\mathcal{Q} \setminus \overline{J^-(\mathcal{I}^+)} \neq \emptyset$. Thus, it follows that we have in fact proven

Theorem 2 *If $\mathcal{Q} \setminus \overline{J^-(\mathcal{I}^+)} \neq \emptyset$, then $\mathcal{I}^+$ is complete and Lemma 3 holds.*

Another point is worth mentioning. It turns out that all statements of this paper except the positivity of mass in $\mathcal{R}$ of Proposition 1 hold equally well if $\mathbf{A}'$ is modified to

$\tilde{\mathbf{A}}'$ In $\mathbf{A}'$, let Γ now be assumed to be an ingoing null segment, let the condition $r = 0$ on Γ be dropped, while let the condition

$$1 - \frac{2m}{r} > 0 \tag{12}$$

on S be added.

(Positivity of mass in $\mathcal{R}$ holds if it is assumed on Γ.)

Given a spherically symmetric initial data for a "reasonable" Einstein-matter system, but where the data now has 2 asymptotically flat ends, let S be a connected piece of the quotient of one of the ends, so that (12) holds, and such that the inward expansion is everywhere negative along S, let p be the endpoint of S, and let Γ be the ingoing null curve in the quotient future development, emanating from p. It is clear that $\tilde{\mathbf{A}}'$ holds for $\mathcal{Q}^+ = J^+(\Gamma \cup S)$.

In particular, $\tilde{\mathbf{A}}'$, $\mathbf{B}'$–$\mathbf{\Sigma T}'$ hold for the region $\mathcal{Q}^+$ defined as above for the development of the Einstein-Maxwell-scalar field system studied in [7, 8, 9].

Finally, it might be useful to point out what we have *not* shown. We have not shown that $i^+ \in \overline{\mathcal{A}'}$, and we have not shown that $\sup_{\mathcal{H}} r = 2\sup_{\mathcal{H}} m$. Both these statement are true, however, in the case of a self-gravitating scalar field.

References

[1] Demetrios Christodoulou *The instability of naked singularities in the gravitational collapse of a scalar field* Ann. of Math. **149** (1999), no 1, 183–217

[2] Demetrios Christodoulou *On the global initial value problem and the issue of singularities* Classical Quantum Gravity **16** (1999), no. 12A, A23–A35

[3] Demetrios Christodoulou *Self-gravitating relativistic fluids: a two-phase model* Arch. Rational Mech. Anal. **130** (1995), no. 4, 343–400

[4] Demetrios Christodoulou *Bounded variation solutions of the spherically symmetric Einstein-scalar field equations* Comm. Pure Appl. Math **46** (1992), no. 8, 1131–1220

[5] Demetrios Christodoulou *The formation of black holes and singularities in spherically symmetric gravitational collapse* Comm. Pure Appl. Math. **44** (1991), no. 3, 339–373

[6] Demetrios Christodoulou *The problem of a self-gravitating scalar field* Comm. Math. Phys. **105** (1986), no. 3, 337–361

[7] Mihalis Dafermos *Stability and Instability of the Cauchy horizon for the spherically-symmetric Einstein-Maxwell-Scalar Field equations* Ann. of Math. **158** (2003), no 3, 875–928

[8] Mihalis Dafermos *The interior of charged black holes and the problem of uniqueness in general relativity* preprint, gr-qc/0307013, to appear in Comm. Pure Appl. Math.

[9] Mihalis Dafermos and Igor Rodnianski *A proof of Price's law for the collapse of a self-gravitating scalar field* gr-qc/0309115, preprint, 2003

[10] Mihalis Dafermos *A note on naked singularities and the collapse of self-gravitating Higgs fields* gr-qc/0403033, preprint, 2004

[11] Mihalis Dafermos and Alan D. Rendall *An extension principle for the Einstein-Vlasov system under spherical symmetry*, in preparation

[12] S. W. Hawking and G. F. R. Ellis *The large scale structure of space-time* Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York, 1973

[13] Edward Malec and Niall Ó Murchadha *Trapped surfaces and the Penrose inequality in spherically symmetric geometries* Phys. Rev. D **49**, no 12, (1994), 6931–4

[14] Roger Penrose *Gravitational collapse and space-time singularities* Phys. Rev. Lett. **14** (1965), 57–59

[15] Roger Penrose *Structure of space-time* Battelle Rencontres, ed. C. M. de Witt and J. A. Wheeler. Benjamin, New York, 1968

[16] Gerhard Rein, Alan Rendall, and Jack Schaeffer *A regularity theorem for solutions of the spherically symmetric Vlasov-Einstein system* Comm. Math. Phys. **168** (1995), 467–478

[17] Alan Rendall, An introduction to the Einstein-Vlasov system *Mathematics of gravitation, Part I (Warsaw 1996)*, 35–68, *Banach Center Publ.* **41** Part 1, 1997